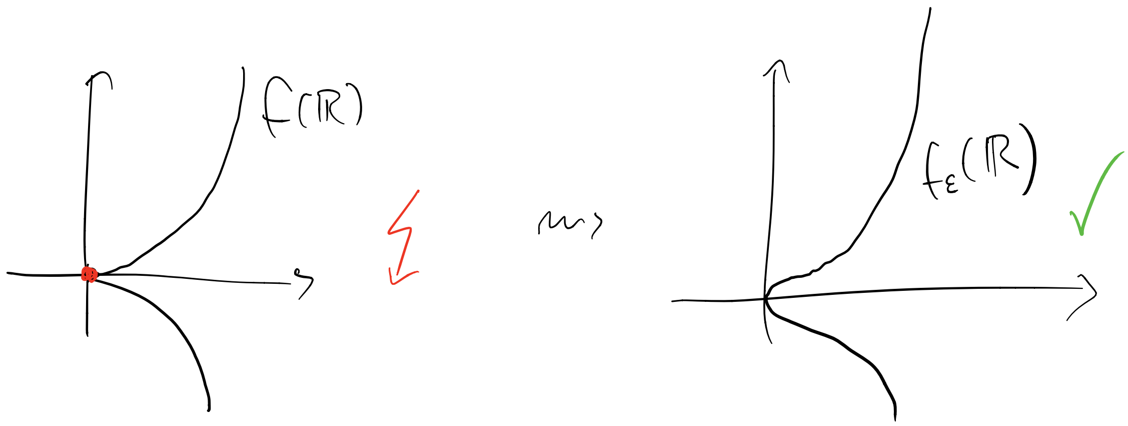
Recap:

- for good aka. 1-generic maps $\Sigma^i(f) \subset M$ smooth submf of M of codim $i(|n-m|+i)$

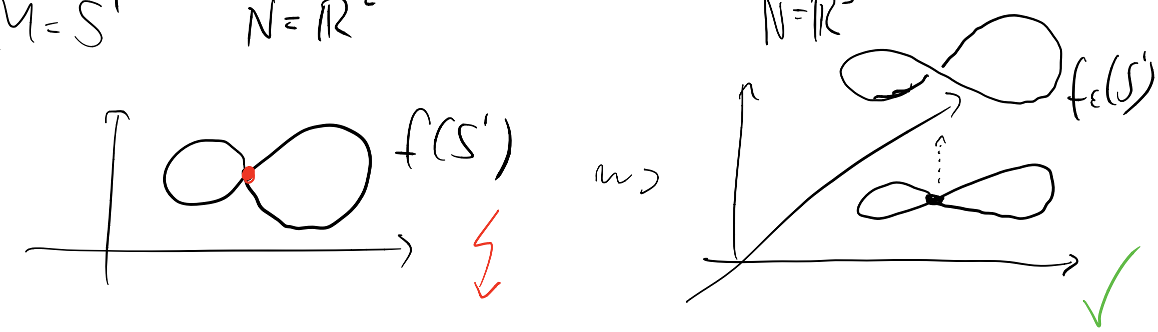
- $\mathrm{Imm}(M,N)$ dense & open in $C^\infty(M,N)$ if $n \geq 2m$

e.g. $M = \mathbb{R}$ $\quad N = \mathbb{R}^2$



- $\mathrm{Emb}(M, \mathbb{R}^{2m+1})$ dense in $C^\infty(M, \mathbb{R}^{2m+1})$

$M = S^1$ $\quad N = \mathbb{R}^2$ $\quad N = \mathbb{R}^3$

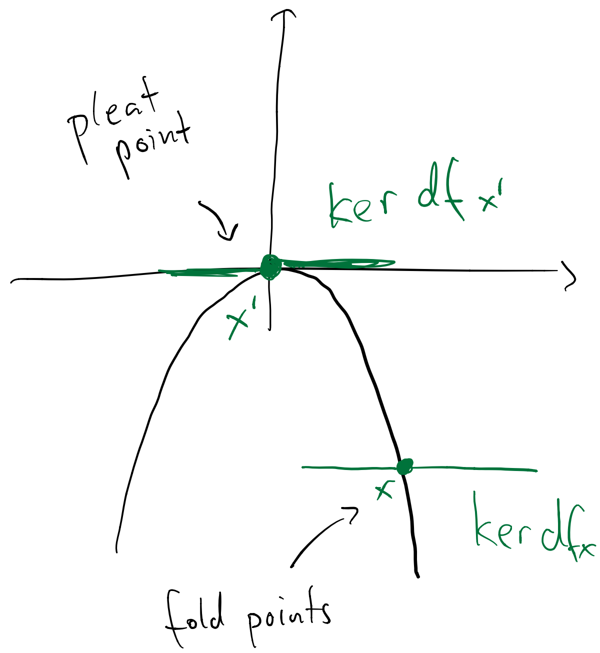


VI Σ^I- classification of singularities

Recall the Whitney map (Exerc. 2, Problems 2&3)

$$w: \mathbb{R}^2 \to \mathbb{R}^2 \quad (x_1, x_2) \mapsto (w_1, w_2) = (x_1^3 + x_2 x_1 \, , \, x_2)$$

$$\mathrm{Crit}(w) = \{ x_2 = -3x_1^2 \} = \Sigma^1(w)$$



idea:

$\Sigma^1(w)$ is smooth mf, consider

$w|_{\Sigma^1(w)}$ and its singularities:

$$w|_{\{x_2 = -3x_1^2\}} : x_1 \longmapsto (-2x_1^3, -3x_1^2)$$

$$\Rightarrow \quad \text{Crit pts} = \{x_1 = 0\}$$

i.e. we can write

$$\Sigma'(w) = \underset{\uparrow \text{ fold pts}}{\Sigma^0(w|_{\Sigma'(w)})} \cup \underset{\uparrow \text{ pleat pt.}}{\Sigma'(w|_{\Sigma'(w)})}$$

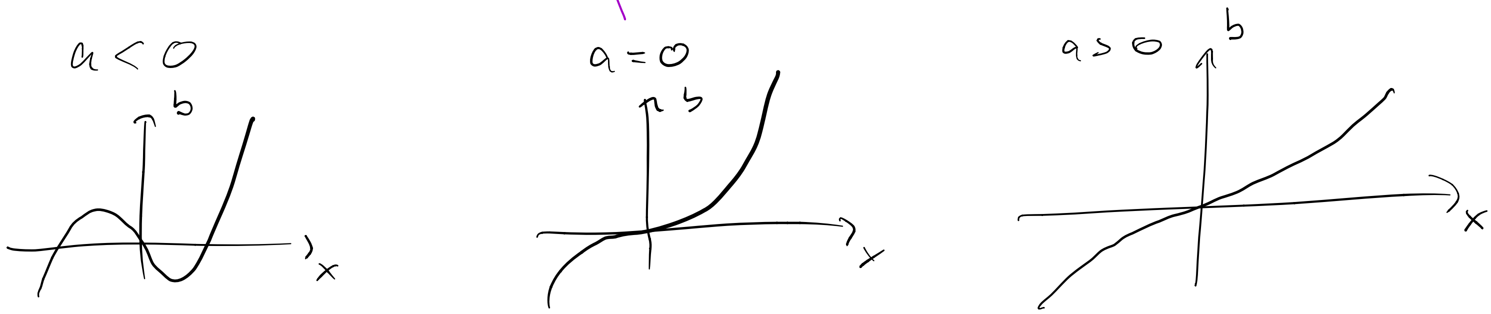
If possible, repeat ...

<u>connection to catastrophe theory:</u>

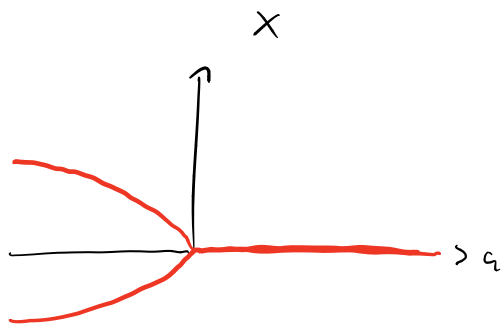
$V = x^4 + ax^2 + bx$ $\qquad$ a,b control parameters

crit pts / singularities: $\frac{dV}{dx} = 4x^3 + 2ax + b \overset{!}{=} 0$

$\Leftrightarrow \quad b = -4x^3 - 2ax$



on $b=0$:



"pitchfork bifurcation"

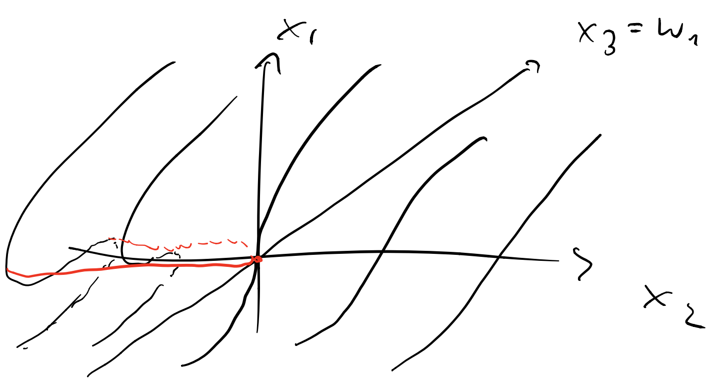
This is equivalent to studying the singularities of w:

Consider the graph $\Gamma(w_1)$

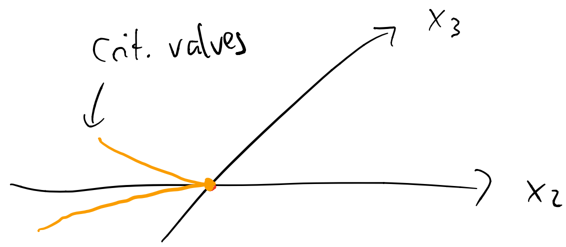
$$\Gamma(w_1) = \left\{ (x_1, x_2, x_1^3 + x_2 x_1) \right\}$$

and its projection to $\mathbb{R}^2$ with coordinates (x_2, x_3)

$$p_w : \Gamma(w_1) \longrightarrow \mathbb{R}^2$$



$$\Big\downarrow p_w$$



we find $\underline{\mathrm{Crit}(p_w)} = \underline{\mathrm{Crit}(w)}$.

For 3-parameter catastrophe

$$V = x^5 + a x^3 + b x^2 + c x$$

$$\frac{dV}{dx} = 5x^4 + 3ax^2 + 2bx + c \stackrel{!}{=} 0$$

"swallowtail catastrophe" (see below)

1. Definition

Let $\Sigma^I(f) = \Sigma^{i_1,\dots,i_k}(f)$ be a smooth mf.
Then define $\Sigma^{i_1,\dots,i_k,i_{k+1}}(f) := \Sigma^{i_{k+1}}\left(f|_{\Sigma^I(f)}\right)$

as the set of points where $\operatorname{corank}\left(f|_{\Sigma^I(f)}\right) = i_{k+1}$.

Q: Is this a well-defined definition?

A:

2. Conj (Thom) / Thm (Boardman)

For a residual set of maps in $C^\infty(M,N)$ this definition makes sense.

Proof (sketch):

1. For I with $I = \{i_1, \dots, i_k\}$ $i_1 \geq i_2 \geq i_3 \dots$ define "appropriate" submanifolds
$$S^I \subset J^k(M,N)$$

2. Call f good / k-generic if
$$j^k f \pitchfork S^I \quad \forall I : |I| = k$$

3. Show that f good $\Rightarrow$ $\Sigma^I(f) = (j^k f)^{-1}(S^I)$

4. Show that
$$T_{S^I} = \{ f \in C^\infty(M,N) \mid j^k f \pitchfork S^I \}$$
is a residual set in $C^\infty(M,N)$. ☐

The hard part is 3., 4. follows then by Thm V.4.

<u>e.g.</u> $f: \mathbb{R}^3 \to \mathbb{R}^3$

$$(x,y,z) \mapsto (x, y, z^4 + xz^2 + yz)$$

$$\mathrm{Crit}(f) = \{ 4z^3 + 2xz + y = 0 \}$$

$$= \Sigma^1(f)$$

$$= \{ (x, -4z^3 - 2xz, z) \mid (x,z) \in \mathbb{R}^2 \}$$

$$\hat{=} \Gamma\big((x,z) \mapsto w_1(x,z) \big)$$

According to Def. 1

$$f_{|\Sigma^1(f)} : (x,z) \mapsto (x, -4z^3 - 2xz, -3z^4 - xz^2)$$

$$df_{|\Sigma^1(f)}(x,z) = \begin{pmatrix} 1 & 0 \\ -2z & -12z^2 - 2x \\ -z^2 & -12z^3 - 2xz \end{pmatrix}$$

$$\Rightarrow \Sigma^{1,1}(f) = \Sigma^1(f_{|\Sigma^1(f)}) = \{ x = -6z^2 \} \subset \Sigma^1(f)$$

So $\Sigma^{1,1}(f) = \{(-6z^2, -16z^3, z) \mid z \in \mathbb{R}\}$

Repeat again.

$$f_{|\Sigma^{1,1}(f)} : z \mapsto (-6z^2, -16z^3, -21z^4)$$

and thus $\Sigma^{1,1,1}(f) = \Sigma^1(f_{|\Sigma^{1,1}(f)}) = \{(0,0,0)\}$.

For maps between 3-mfs this is the generic case, i.e. only singularities of type $\Sigma^1, \Sigma^{1,1}, \Sigma^{1,1,1}$ occur.

In dimension 4 a $\Sigma^{2,0}$ appears (see exercise)!

However,

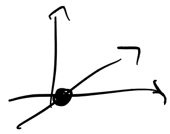
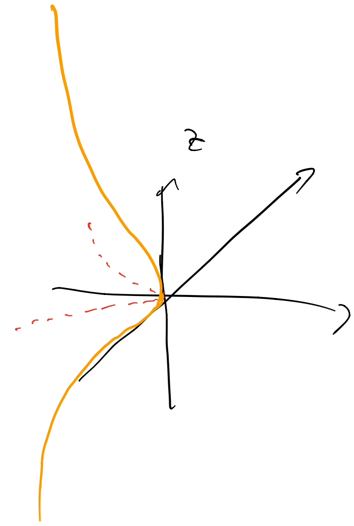
3. Thm (Morin)

Let $m=n$ and $f: M \to N$ good. Then for $x_0 \in \Sigma^{\underbrace{1,1,\dots,1}_{k}}(f)$ there exists coordinates s.t. locally f is given by

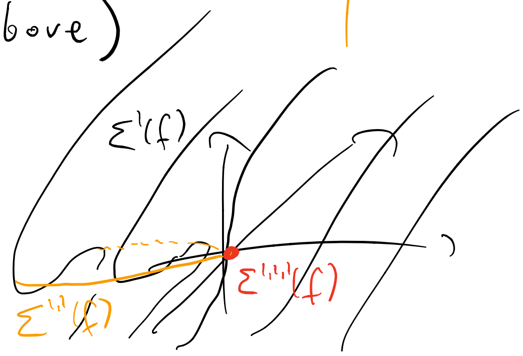
$$(x_1, \dots, x_m) \mapsto (x_1, \dots, x_{m-1}, x_m^{k+1} + x_m^{k-1} \cdot x_{k-1} + \dots + x_m^2 \cdot x_2 + x_m \cdot x_1)$$

How to visualize / where is the swallowtail?

1. As $\Gamma(w_1)$

$\Sigma^{1,1,1}(f) =$  $\Sigma^{1,1}(f) =$ 

$\Sigma^1(f) \cong \Gamma(w_1)$ (see above)

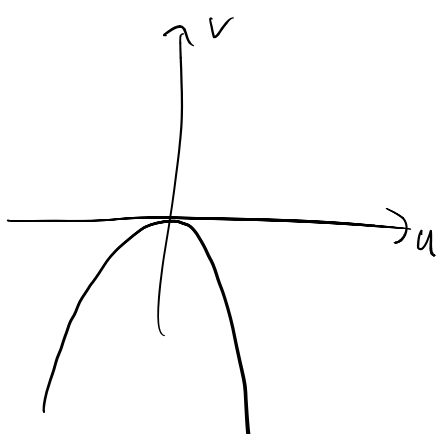


2. Critical <u>values</u> of f

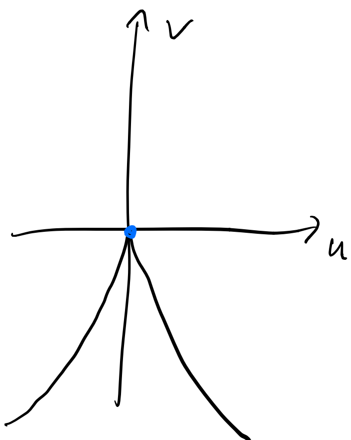
$$f(\Sigma^1(f)) = \mathrm{im}(f_{|\Sigma^1(f)})$$

$$= \{(x, \underbrace{-4z^3-2xz}_{=:u}, \underbrace{-3z^4-xz^2}_{=:v})\}$$

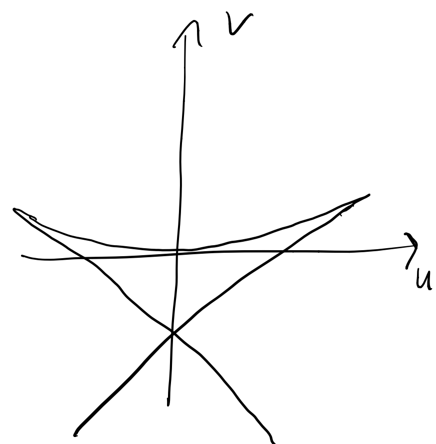
$x > 0$



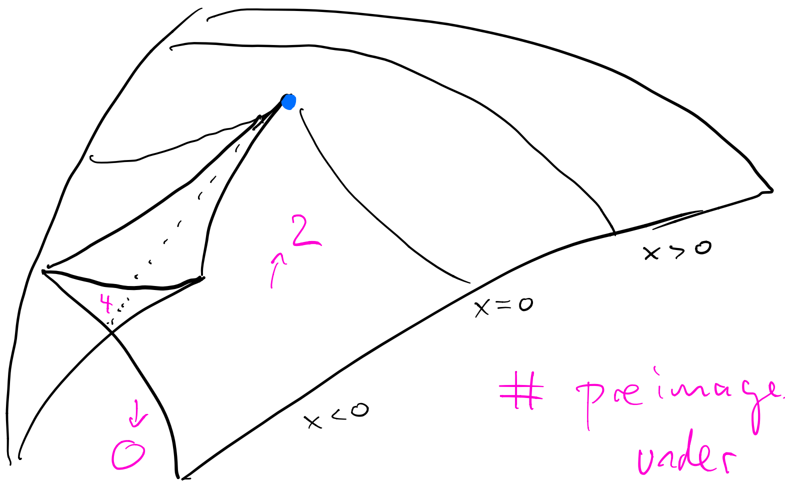
$x = 0$



$x < 0$



put together to get the swallowtail surface



preimages of pts under f